Applied Time Series Analysis
FS 2012 – Week 12

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Spectral Analysis

Idea: Time series are interpreted as a combination of cyclic components, and thus, a linear combination of harmonic oscillations.

Why: As a descriptive means, showing the character and the dependency structure within the series.

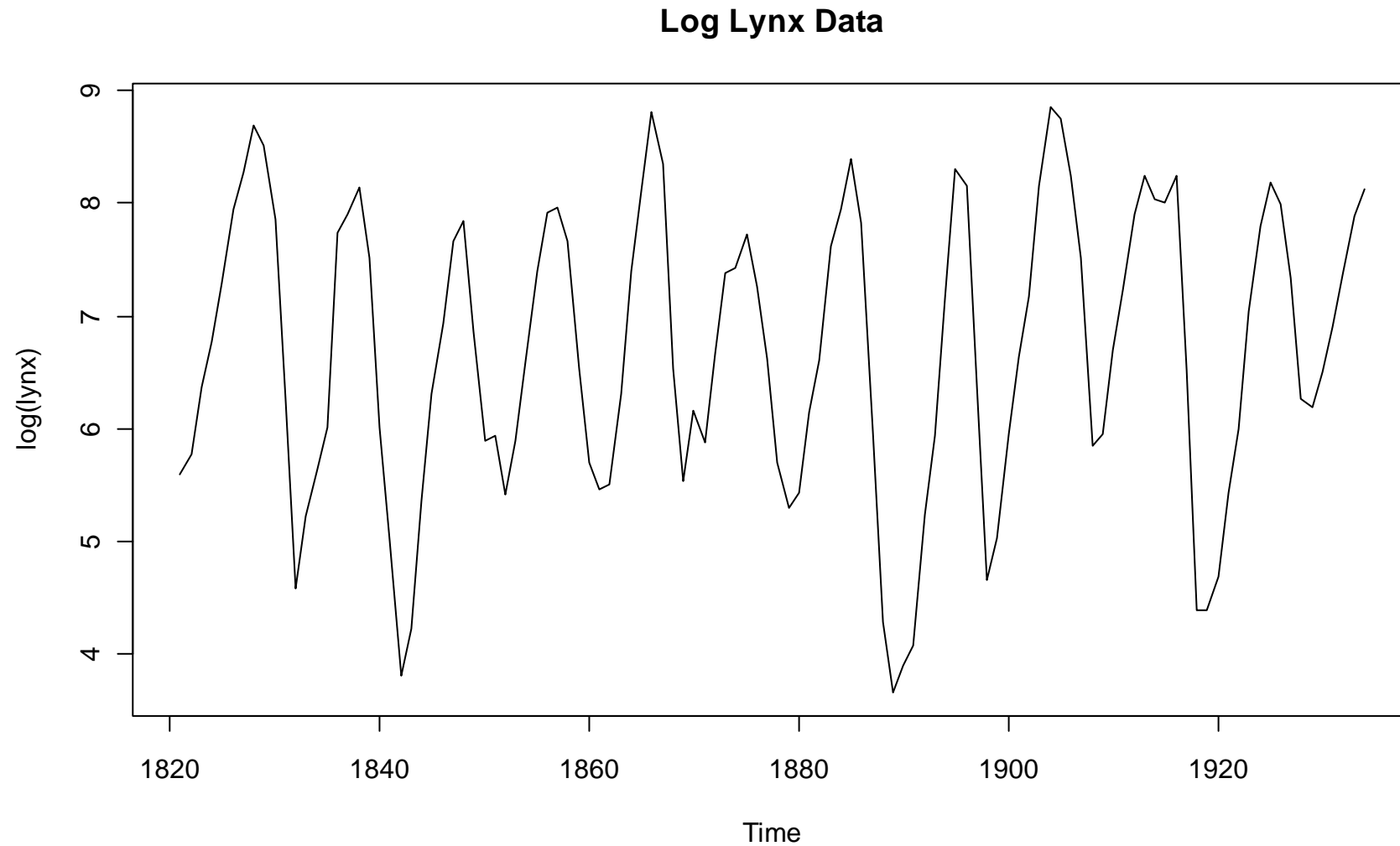**What**:   It is in spirit, but also mathematically, closely related to the correlogram

Where: - engineering
- economics
- biology/medicine

Lynx Data

Log Lynx Data

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Ocean Wave Data

Ocean Wave Height Data, Part 1
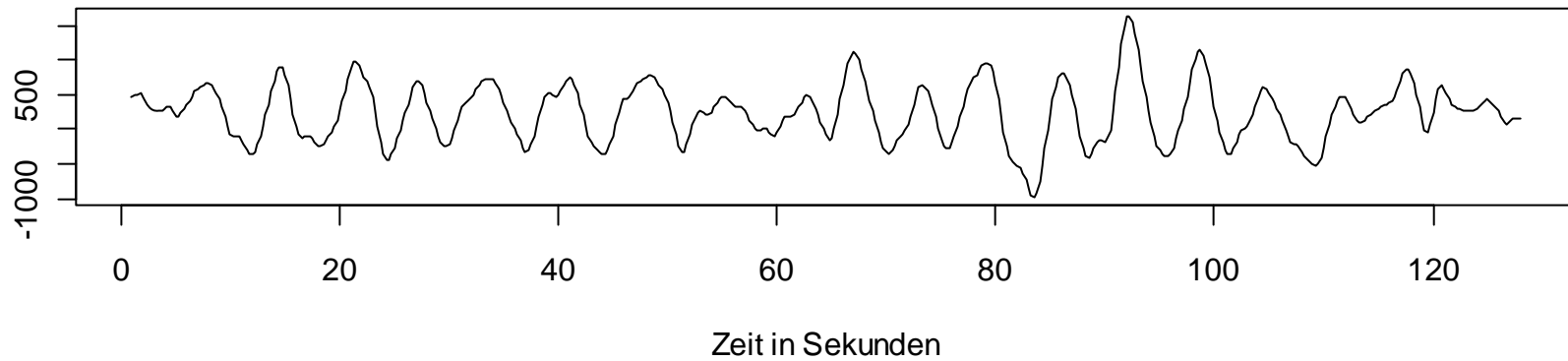
Zeit in Sekunden

Ocean Wave Height Data, Part 2

Zeit in Sekunden
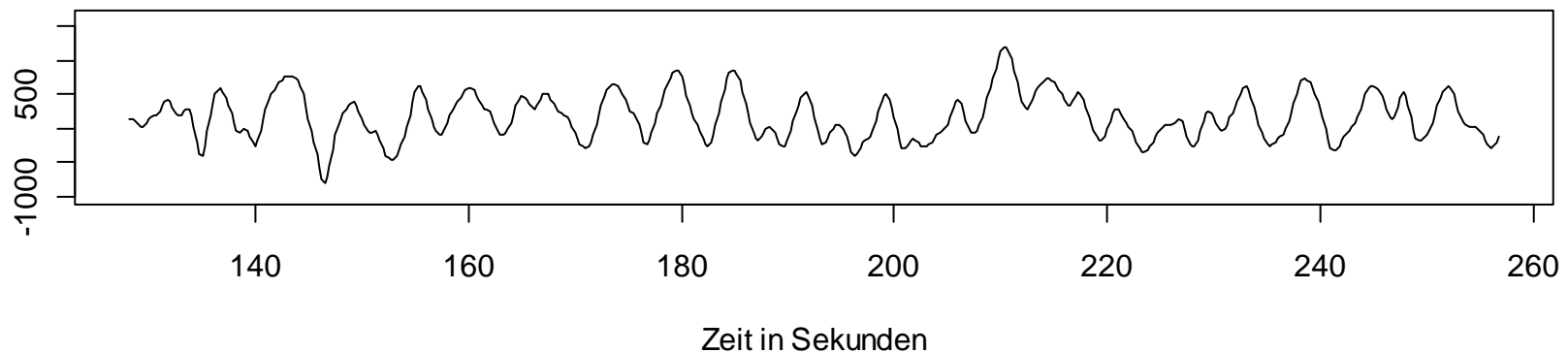
2-Component-Mixture Data

2-Component-Mixture: Series 1
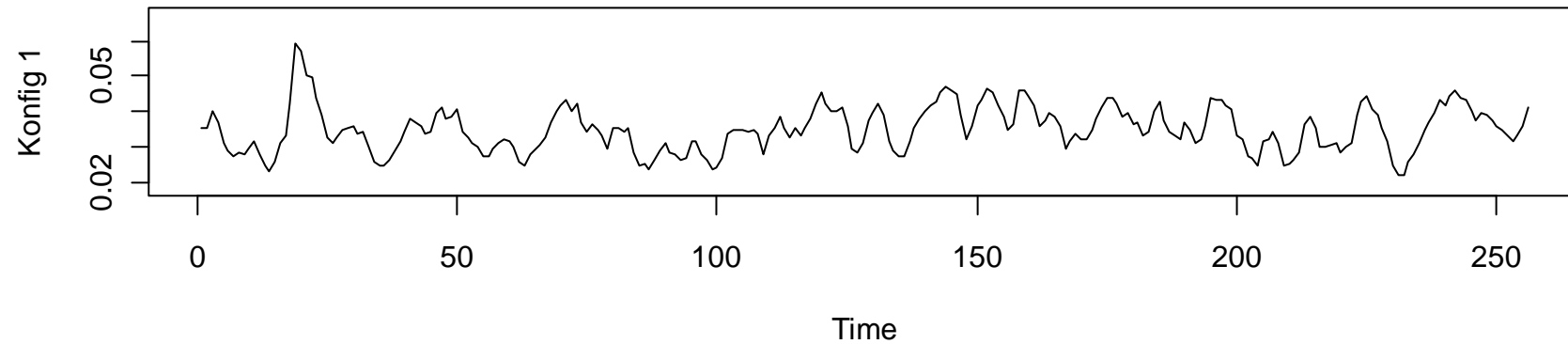
2-Component-Mixture: Series 2
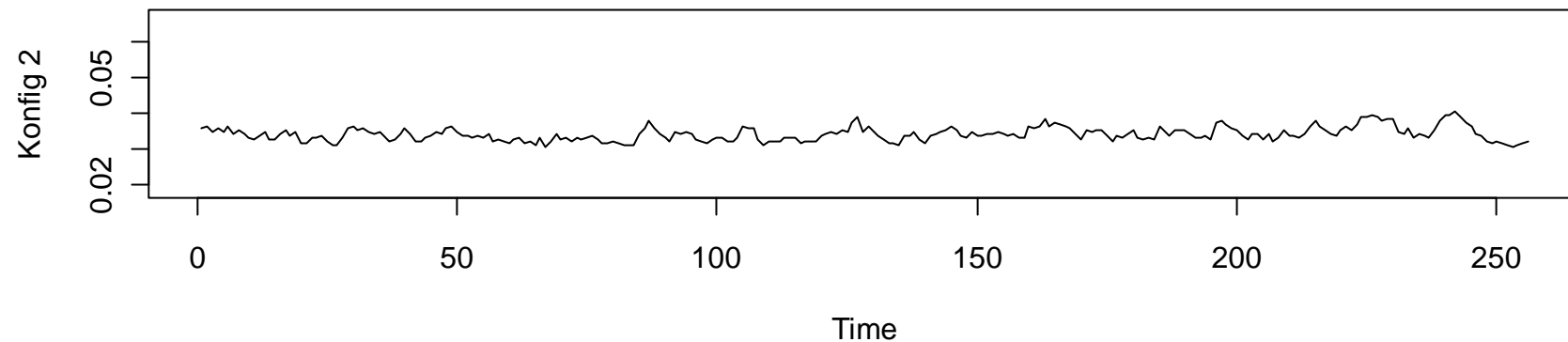
Harmonic Oscillations

The most simple periodic functions are sine and cosine, which we will use as the basis of our analysis.

A harmonic oscillation has the following form:

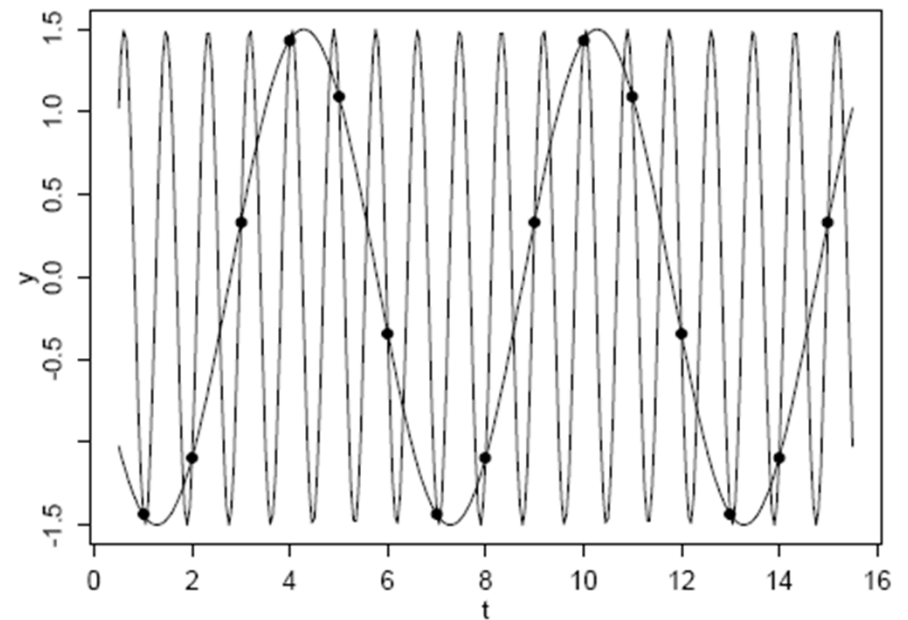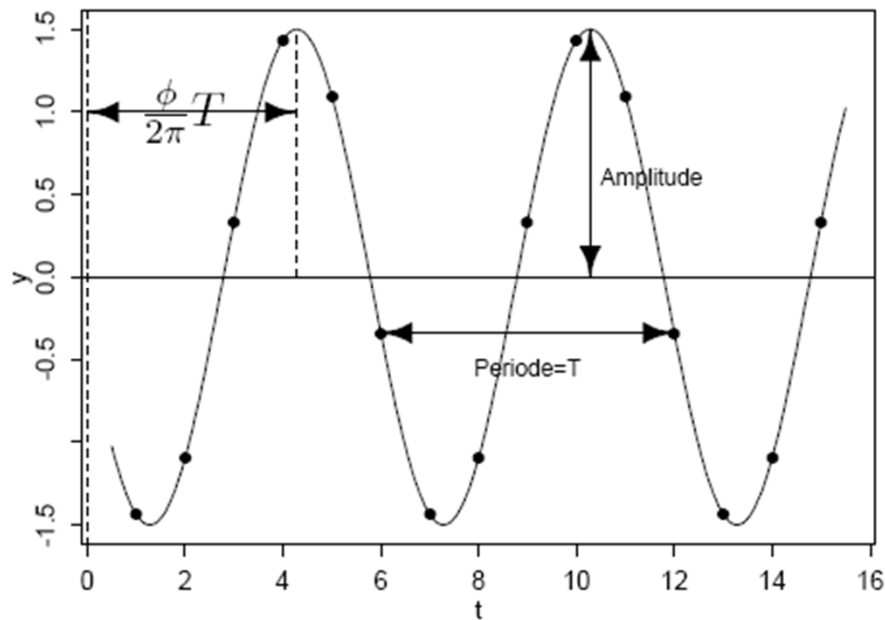$$y(t) = \alpha \cos(2\pi\nu t) + \beta \sin(2\pi\nu t)$$

For the derivation, **see the blackboard…**

- In discrete time, we have aliasing, i.e. some frequencies cannot be distinguished ($\rightarrow$ see next slide).

- The periodic analysis is limited to frequencies between 0 and 0.5, i.e. things we observe at least twice.

Aliasing

Regression Model & Periodogram

We try to write a time series with a regression equation containing sine and cosine terms at the fourier frequencies.
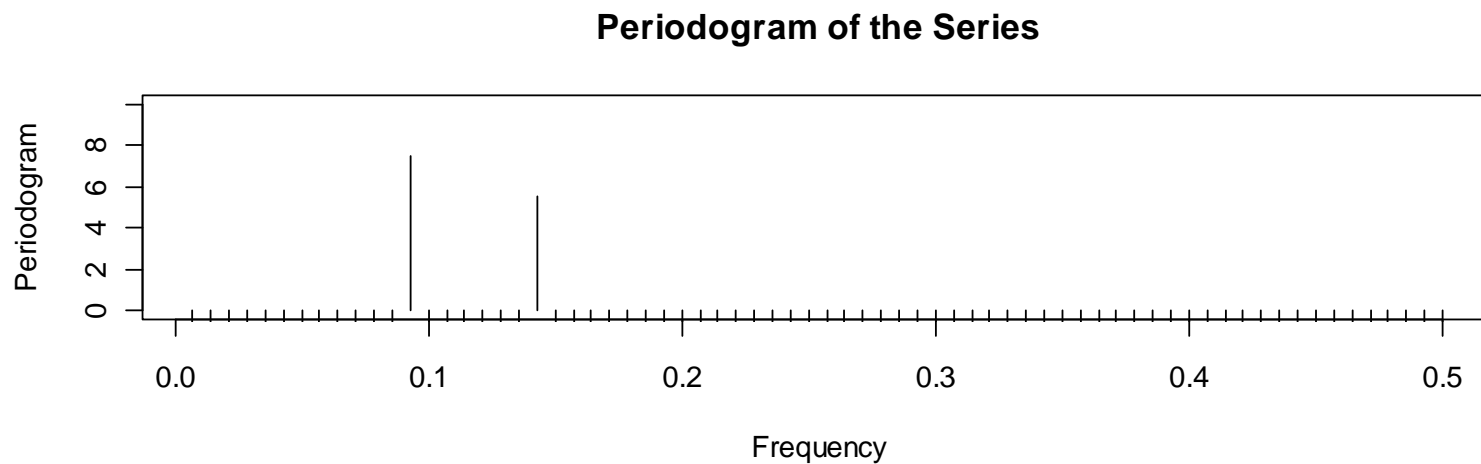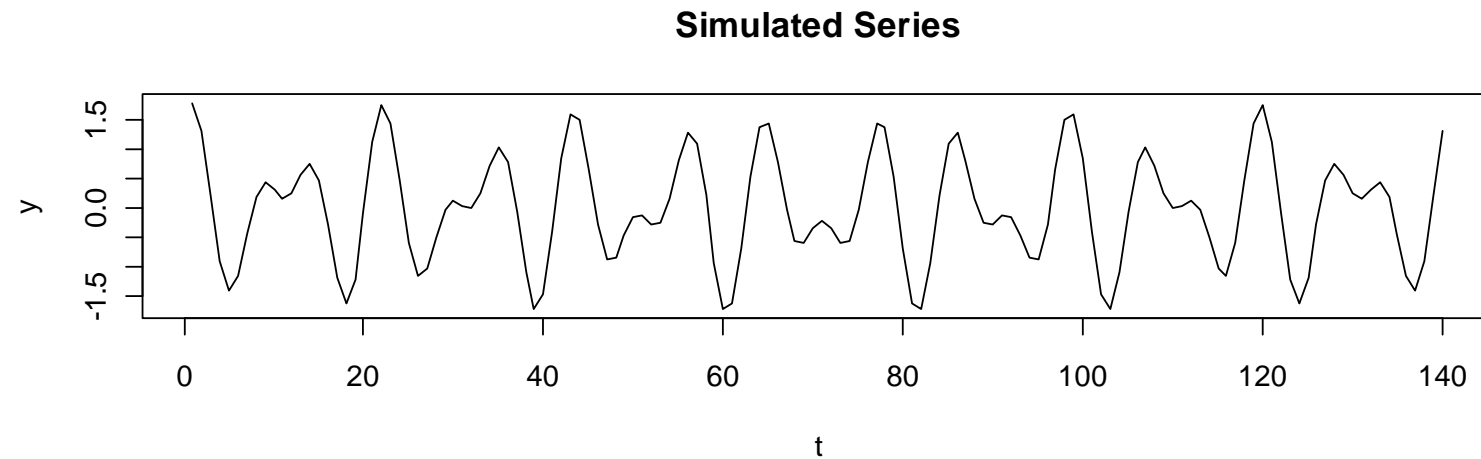
→ **see the blackboard**

The most important frequencies within the series, which when omitted, lead to pronounced increase in goodness-of-fit.

- This idea is used as a proxy for the periodogram,
  → **see the blackboard…**

- However, if the „true" frequency is not a fourier frequency, we have leakage (→ see next 2 slides).

Periodogram of a Simulated Series

Simulated Series

Periodogram of the Series

Periodogram of the Shortened Series

Periodogram of the Shortened Series
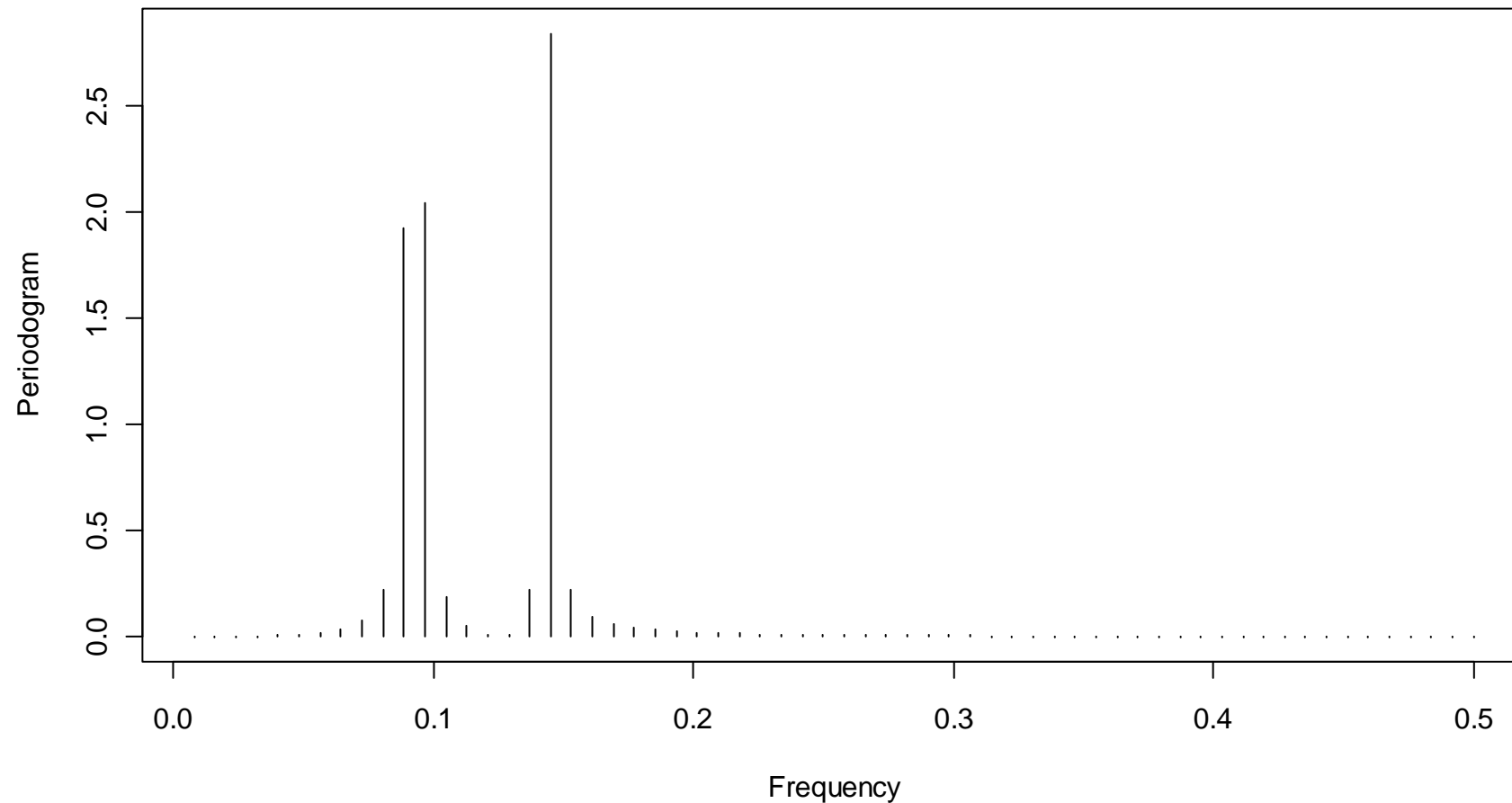
Properties of the Periodogram

Periodogram and correlogram are mathematically equivalent, the former is the **fourier transform** of the latter.
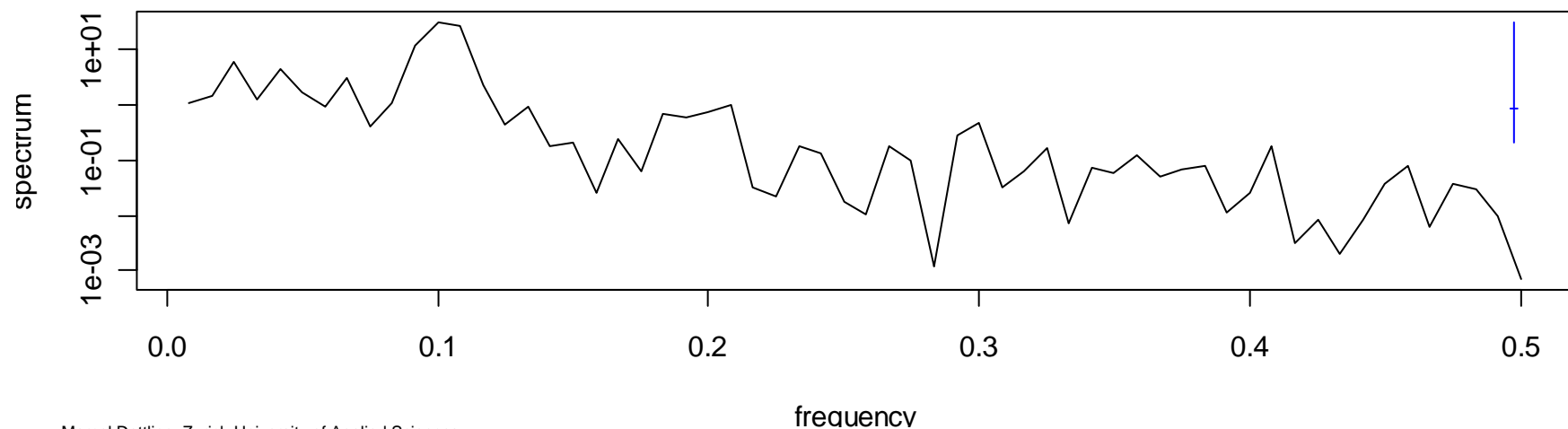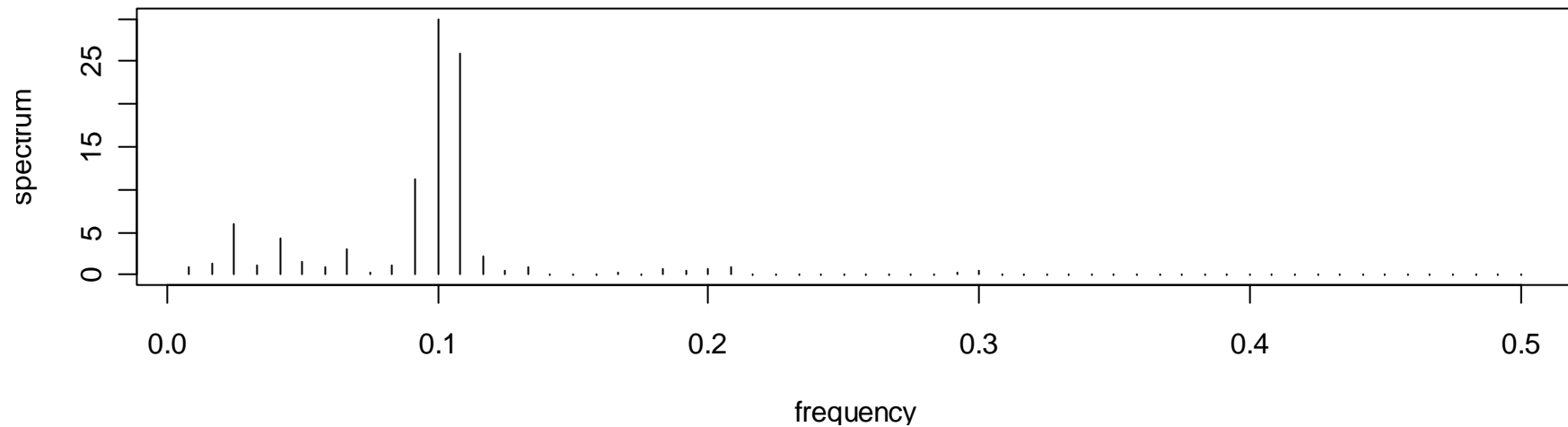
→ **see the blackboard** for the derivation

Note: this is a reason why we divided by 1/n in the ACV.

- $I(v_k)$ or $\log(I(v_k))$ are plotted against k/n

- Estimates seem rather instable and noisy

- On the log-scale, most frequencies are present

- It seems as if smoothing is required for interpretation.

Periodogram of the Log Lynx Data

Periodogram of the Ocean Wave Data

Periodogram of the Ocean Wave Data
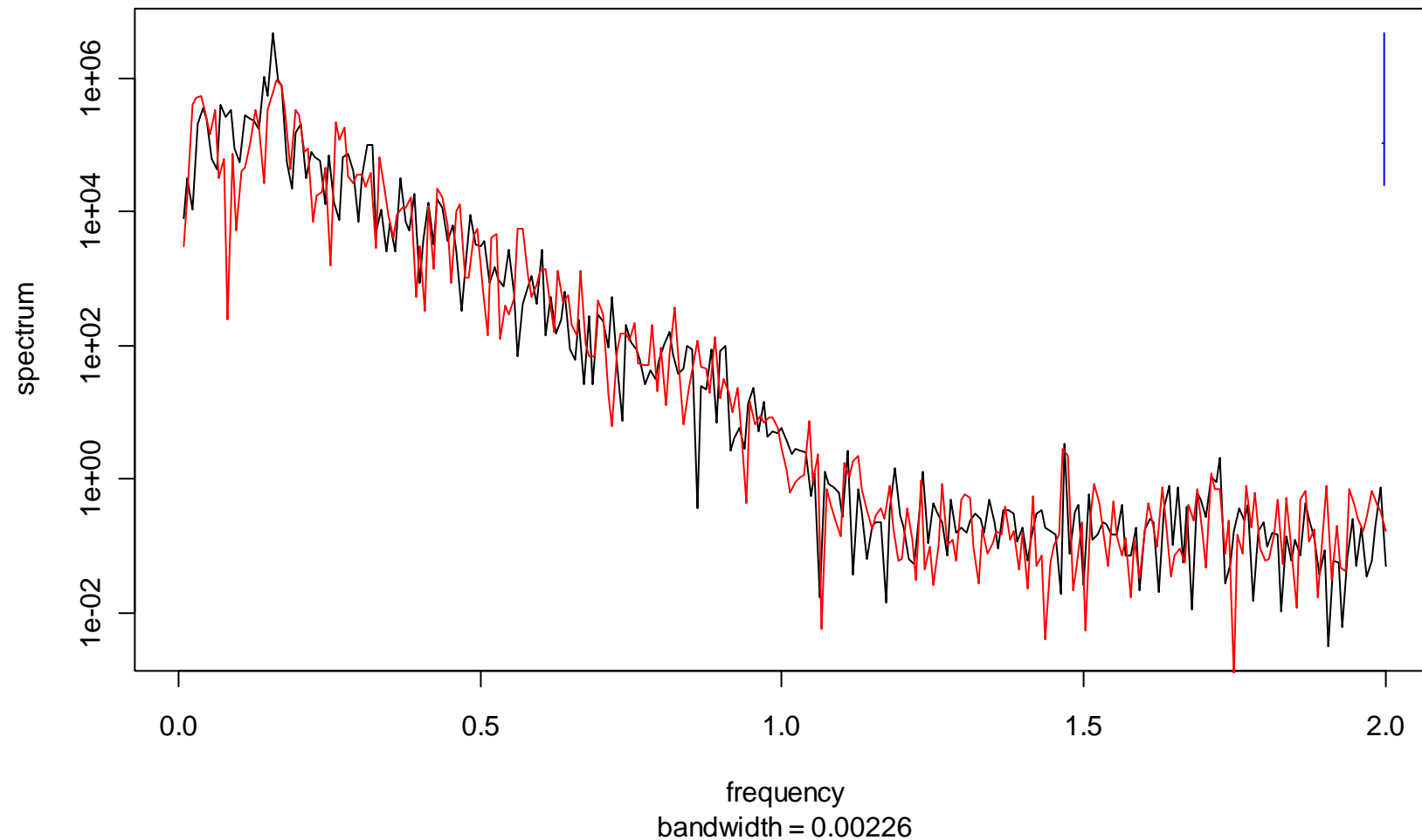
Periodogram of the 2-Component-Mixture
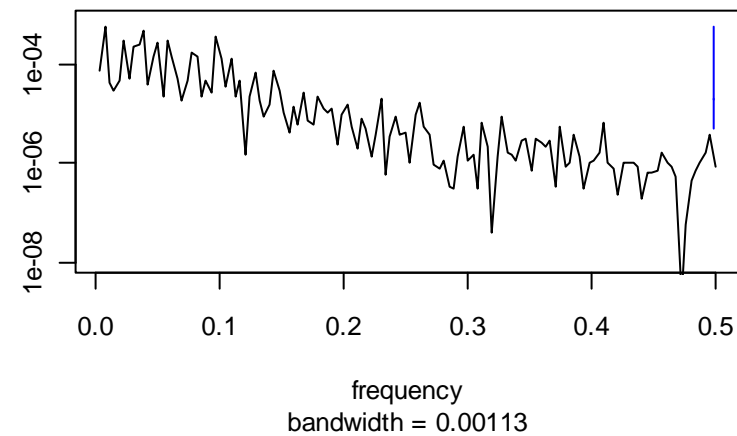

2-Component-Mixture: Config 1
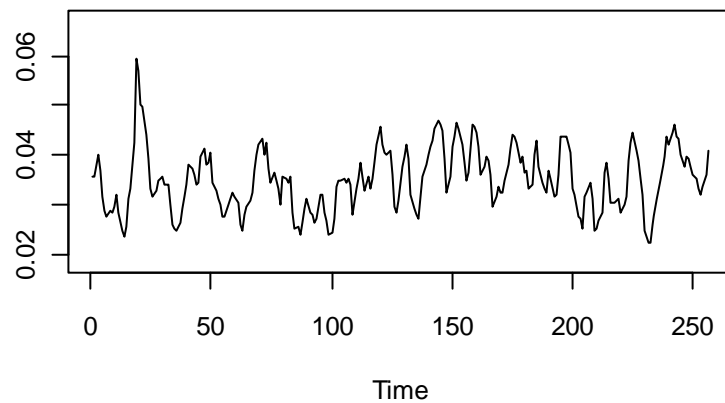
Periodogram of Config 1

2-Component-Mixture: Config 2
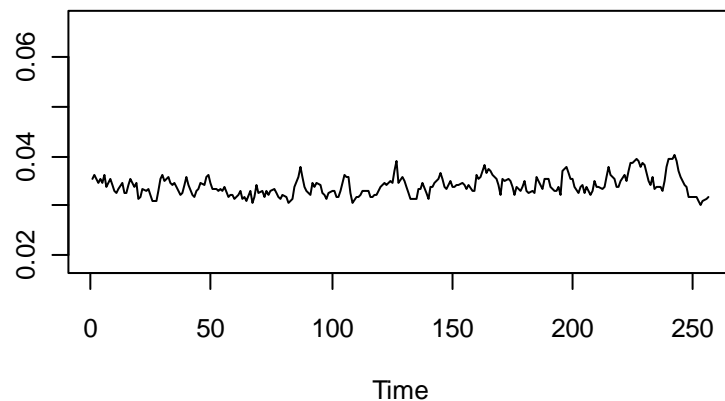
Periodogram of Config 2
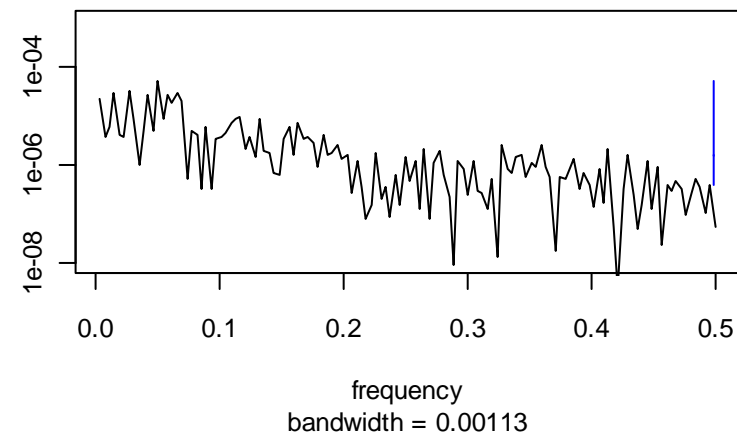
The Spectrum

Observed time series	⇔	Stochastic process
Empirical ACF	⇔	Theoretical ACF
Periodogram	⇔	Spectrum

There is a link between ACF and periodogram/spectrum

$$f(v) = \sum_{k=-\infty}^{+\infty} \gamma(k)\cos(2\pi v k)$$

and

$$\gamma(k) = \int_{-0.5}^{+0.5} f(v)\cos(2\pi v k)dv$$

respectively. The spectrum is thus the Fourier transformation of the ACV.

What's the Spectrum Good For?

Theorem: **Cramer Representation**

Every stationary process can be written as the limit of a linear combination consisting of harmonic oscillations with random, uncorrelated amplitudes.

- The spectrum characterizes the variance of all these random amplitudes.

- Or vice versa: $\int_{v_1}^{v_2} f(v)\,dv$ is the variance between the frequencies that make the integration limits.

- The spectrum takes only positive values. Thus, not every ACF sequence defines a stationary series.

A Few Particular Spectra

- **White noise**

  → the spectrum is constant over all frequencies.

- **AR(1),** see next slide

  → already quite a complicated function α_1

- **ARMA (p,q)**

  → the characteristic polynoms determine the spectrum

  $$f(\nu) = \sigma_E^2 \frac{|\Theta(\exp(-i2\pi\nu))|}{|\Phi(\exp(-i2\pi\nu))|}$$

- Note: to generate m maxima in the spectrum, we require an AR-model, where the order is at least $2m$.

Spectrum of AR(1)-Processes

Spectrum of Simulated AR(1)-Processes
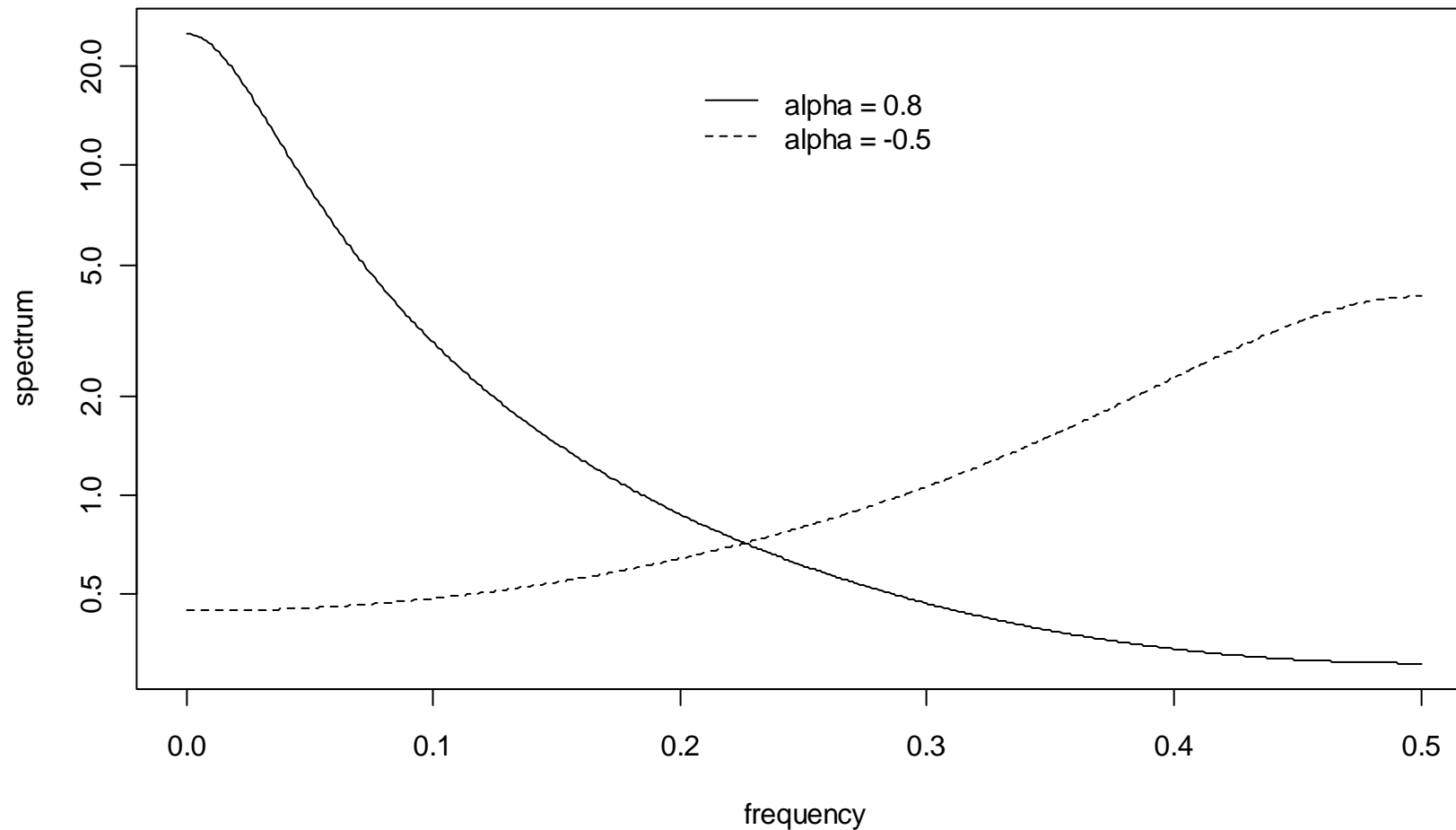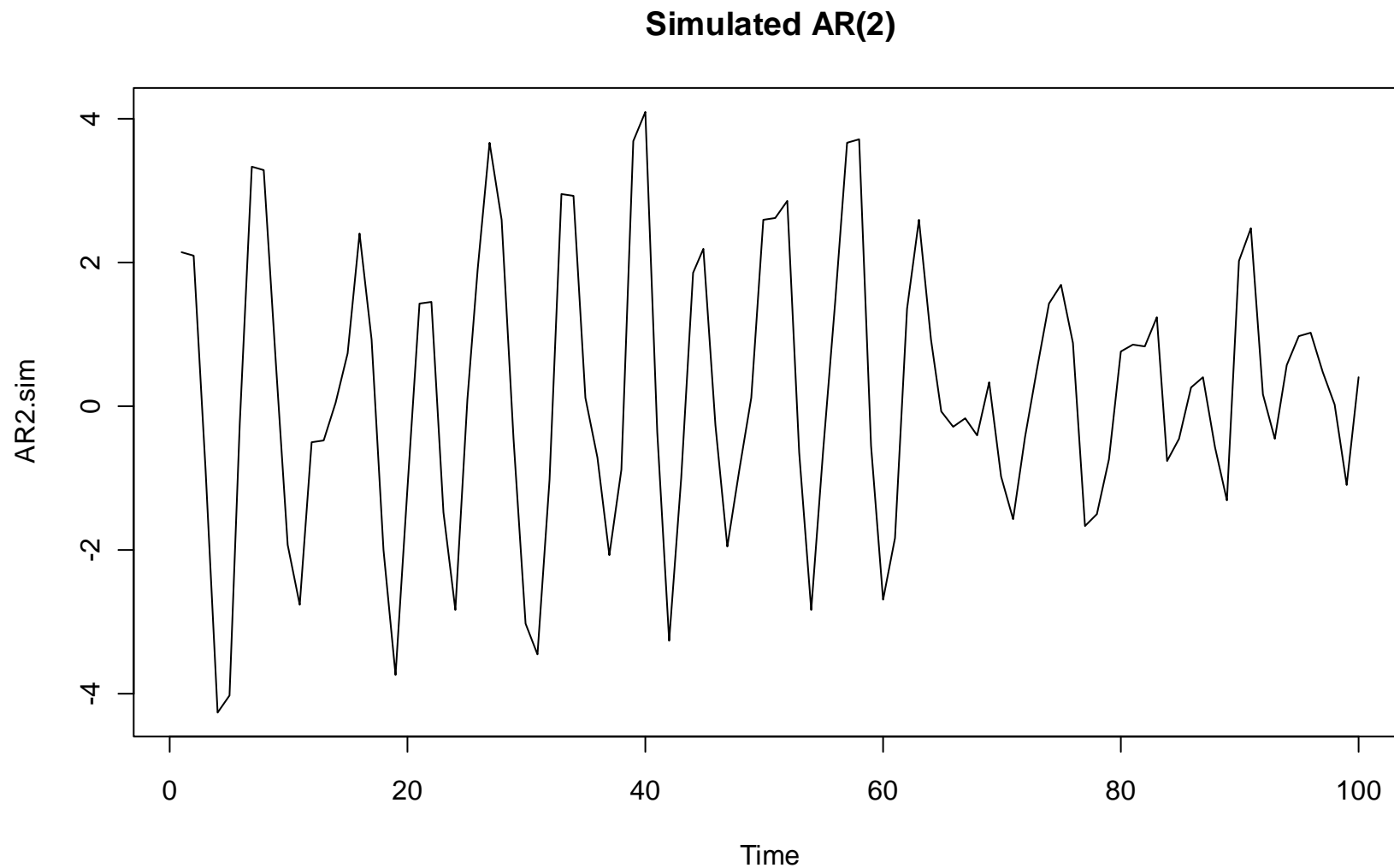
Simulated AR(2)-Process

Simulated AR(2)

ACF/Spectrum of Simulated AR(2)-Process
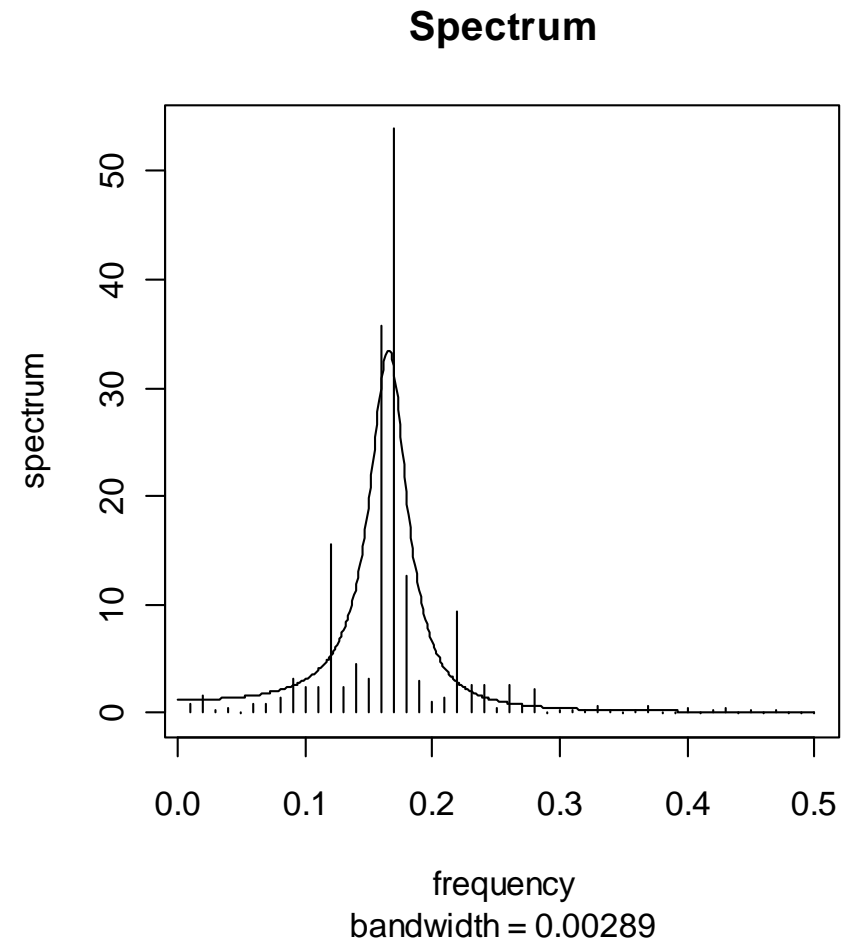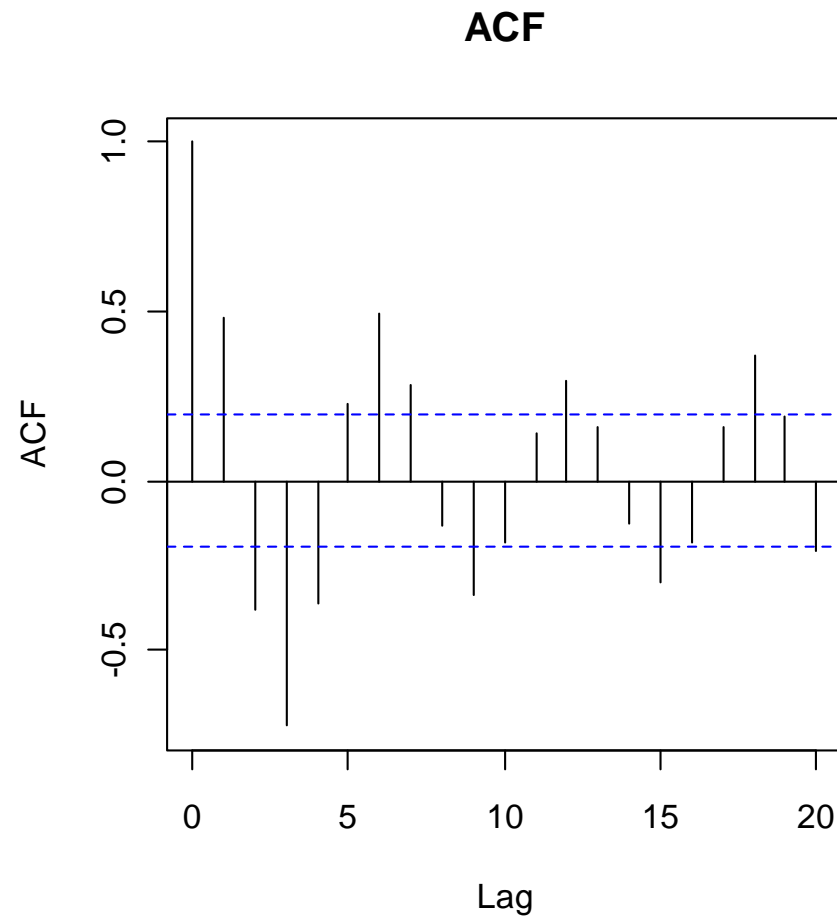
Spectral Analysis

- **Spectral analysis** is a descriptive technique, where the time series is interpreted as a linear combination of harmonic oscillations.

- The **periodogram** shows empirically, which frequencies are „important", i.e. lead to a substantial increase in RSS when ommitted from the linear combination.

- The **spectrum** is the theoretical counterpart to the periodogram. It can also be seen as the Fourier transformation of the theoretical autocovariances.

- The periodogram is a poor **estimator** for the spectrum: it's not smooth and inconsistent.

Improving the Raw Periodogram

1) Smoothing with a running mean

+   simple approach

-        choice of the bandwith

2) Smoothing with a weighted running mean

+   choice of the bandwith is less critical

-        difficulties shift to the choice of weights

3) Weighted plug-in estimation

+   weighted Fourier trsf. of estimated autocovariances

-        choice of weights

4) Piecewise periodogram estimation with averaging

+   can serve as a check for stationarity, too

Improving the Raw Periodogram

5) Spectrum of an estimated model

+ fundamentally different from 1)-4)

- only works for „small" orders p

6) Tapering

+ further modification of periodogram estimation

+ reduces the bias in the periodogram

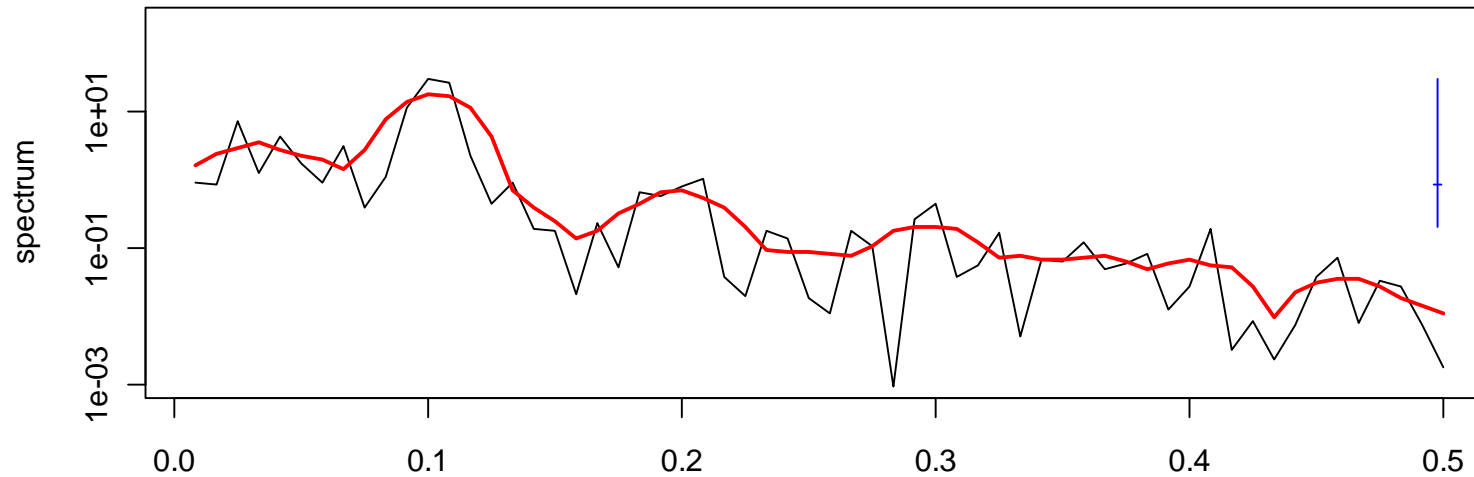+ should always be applied

7) Prewhitening and Rescoloring

+ model fit and periodogram estimation on residuals
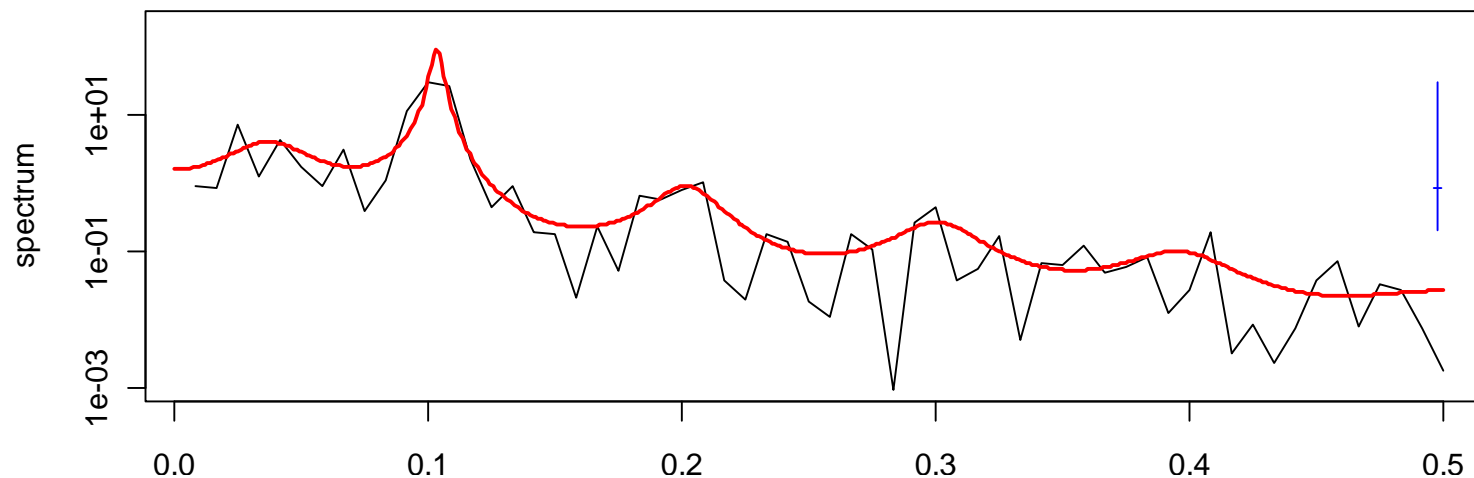
+ the effect of the model will be added again

Modified Periodogram of log(Lynx) Data

Modified Periodogram of log(Lynx) Data

Piecewise periodogram of ocean wave data

frequency
bandwidth = 0.00977