More about Single Factor Experiments

Parameter estimation

- Effect Model (1):

$$Y_{ij} = \mu + A_i + \epsilon_{ij}, \qquad \sum J_i A_i = 0$$

Estimation: $\widehat{\mu + A_i} = y_{i.} \quad \hat{\mu} = y_{..} \quad \hat{A}_i = y_{i.} - y_{..}$

- Effect Modell (2):

$$Y_{ij} = \mu + A_i + \epsilon_{ij}, \qquad A_1 = 0$$

Estimation: $\hat{\mu} = y_{1.} \quad \hat{A}_i = y_{i.} - y_{1.}$

- Mean Model: $Y_{ij} = \mu_i + \epsilon_{ij}$ Estimation: $\hat{\mu}_i = y_{i.}$

Remarks:

- To interpret parameters correctly you must know which model has been used. Coefficients have different meanings.
- Prediction and residuals are always the same.
  Prediction:
  $$\hat{y}_{ij} = \begin{cases} \hat{\mu} + \hat{A}_i \\ \hat{\mu}_i \end{cases} = y_{i.}$$
  Residual:
  $$r_{ij} = y_{ij} - y_{i.}$$

Anova and Regression

- Analysis of variance models can be written as multiple regression models with indicator variables.
- Analysis of variance models are more intuitiv.
- Parameter estimators $y_{..}$, $y_{i.}, \ldots$ are Least Squares estimators.

Berliner Pfannkuchen

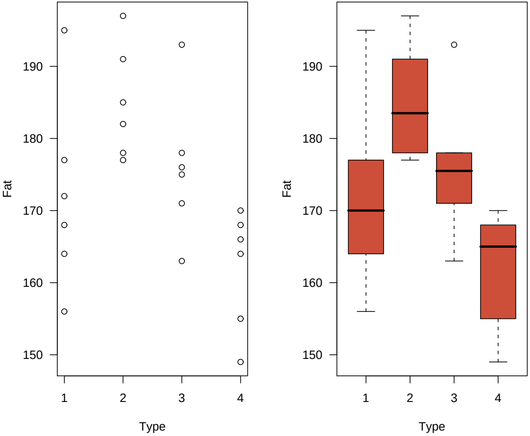
Data

Response: Fat absorption of 24 Berliner [g]

Type of Fat	Fat Absorption						Mean
1	164	172	168	177	156	195	172.0
2	178	191	197	182	185	177	185.0
3	175	193	178	171	163	176	176.0
4	155	166	149	164	170	168	162.0

balanced design: equal replication

Graphical display



R: Anova table

```
> mod2=aov(fat~type,data=berliner)
> summary(mod2)
            Df Sum Sq Mean Sq F value Pr(>F)
type         3 1636.5   545.5  5.4063 0.0069**
Residuals   20 2018.0   100.9

> coef(mod2)
(Intercept)    type2       type3       type4
        172      13           4         -10
```

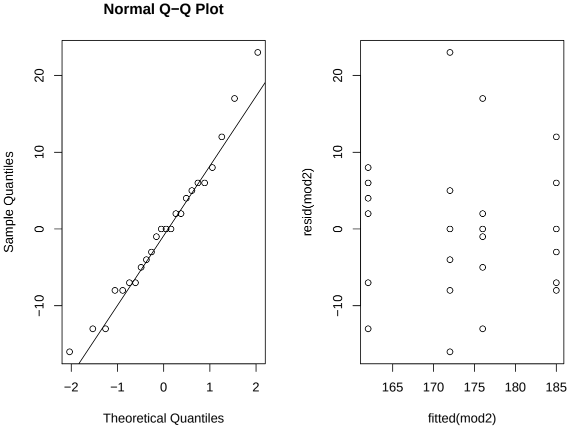
Question: What do these coefficients mean?
command model.matrix() can be used to see the design matrix

Model checking

Modell: $Y_{ij} = \mu + A_i + \epsilon_{ij}, \quad \epsilon_{ij} \sim N(0, \sigma^2)$ i.i.d.

- Normal plot of residuals $r_{ij} = y_{ij} - y_{i.}$ To detect Outliers.
  Normal distribution not crucial in randomized experiments.
  Nonparametric test: Kruskal-Wallis
- Equal variances: Plot r_{ij} vs $y_{i.}$
  $\sigma^2_{min} < \frac{1}{9}\sigma^2_{max}$ (balanced designs), log-$\sqrt{}$ -transformation, weights
- Independent observations: Plot r_{ij} vs time, order
  more complex model, analysis

Residual plots



Treatment differences

F test significant $\Longrightarrow$ There are treatment effects.
Which? How large are the effects?

Treatment differences estimated by $y_{i\cdot} - y_{i'\cdot}$.

Fat type 2 − Fat type 1: $185 - 172 = 13$
Fat type 3 − Fat type 1: $176 - 172 = 4$
Fat type 4 − Fat type 1: $162 - 172 = -10$

Standard error of a treatment difference:
$\sqrt{\sigma^2(1/J + 1/J)} = \sqrt{2\sigma^2/J}$, estimated by $\sqrt{2MS_{res}/J}$.

Example: $\sqrt{2 \cdot 100.9/6} = 5.799$

Are Type 2 and 1 significantly different?

t test for $H_0 : A_2 = A_1$

$$t = \frac{y_{2.} - y_{1.}}{\sqrt{2MS_{res}/J}} = \frac{13}{5.799} = 2.242 > 2.086 = t_{0.975,20}, p = 0.036$$

Confidence interval for Type 2 - Type 1:

$$13 \pm 2.086 \cdot 5.799 = 13 \pm \underbrace{12.097}_{LSD} = (0.9, 25.1)$$

Multiple pairwise comparisons

Are all pairs of treatments different? Problem: α_E increases.

- Bonferroni correction for 6 pairwise comparisons:
  Significance level: $\alpha_T = 0.05/6 = 0.0083$
  Critical value: $t_{1-0.05/2\cdot 6,20} = 2.927$
  Difference between Type 2 and 1 not significant.
- Dunnett's method for multiple comparisons with a control group.
- Tukey method for pairwise comparisons:
  critical values for the distribution of $\max |y_{i.} - y_{i'.}|$

Tukey method

Reject $H_0 : A_2 = A_1$, if

$$|t| > \frac{1}{\sqrt{2}} q_{1-\alpha,I,N-I}$$

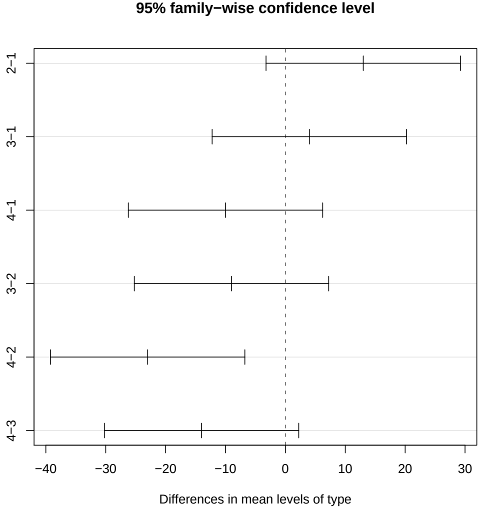
with $q_{...}$ the quantile of the Studentized Range distribution.

Example: $|t| > \frac{3.958}{\sqrt{2}} = 2.799$.
Type 2 and 1 do not differ significantly.

Tukey Confidence interval for Type 2 - Type 1:

$$13 \pm 2.799 \cdot 5.799 = 13 \pm \underbrace{16.23}_{HSD} = (-3.2, 29.2)$$

R: plot(TukeyHSD(mod2,“type“))



Complex comparisons

Is there a difference between fat types 1 and 4 vs 2 and 3?

$$H_0 : \frac{A_1 + A_4}{2} = \frac{A_2 + A_3}{2} \Leftrightarrow \frac{A_1}{2} - \frac{A_2}{2} - \frac{A_3}{2} + \frac{A_4}{2} = 0$$

Hypotheses can be written as linear combinations $\sum \lambda_i A_i$.

Question: What about the question before: is there a difference between type 1 and type 2?

$$H_0 : A_1 - A_2 = 0$$

Contrasts

Contrast:

$$C = \sum_{i=1}^{I} \lambda_i A_i \text{ with } \sum \lambda_i = 0$$

Ex1: $C_1 = (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$ Ex2: $C_2 = (1, -1, 0, 0)$

Ex2?

C can be estimated by

$$\begin{aligned} \hat{C} &= \sum \lambda_i \hat{A}_i = \sum \lambda_i (y_{i.} - y_{..}) \\ &= \sum \lambda_i y_{i.} - y_{..} \sum \lambda_i = \sum \lambda_i y_{i.}. \end{aligned}$$

Testing of a contrast

Reject $H_0 : \sum \lambda_i A_i = 0$, if

$$|t| = |\frac{\hat{C}}{\sqrt{MS_{res} \sum \frac{\lambda_i^2}{J_i}}}| > t_{0.975, N-I}$$

Equivalently,if

$$F = t^2 = \frac{\hat{C}^2 / \sum \lambda_i^2 / J_i}{MS_{res}} = \frac{SS_C}{MS_{res}} > F_{0.95,1,N-I}$$

SS_C denotes the sum of squares of the contrast C.

Orthogonal contrasts

- There are $I - 1$ linearly independent contrasts.
- Two contrasts $C_1 = \sum \lambda_i A_i$ and $C_2 = \sum \lambda'_i A_i$ are called orthogonal, if $\sum \lambda_i \lambda'_i = 0$.
- It is always possible to find $I - 1$ orthogonal contrasts.

Partitioning of Treatment Sum of Squares

(only balanced designs)

orthogonal contrasts $\longrightarrow$ uncorrelated estimates $\longrightarrow$
t tests nearly independent

$SS_C = J\hat{C}^2/\sum\lambda_i^2$ sum of squares of the contrast C

If $C_1, C_2, \ldots, C_{I-1}$ are orthogonal contrasts, then

$$SS_{treat} = SS_{C_1} + SS_{C_2} + \cdots + SS_{C_{I-1}}$$

Recommendation: Multiple Comparison

n planned , orthogonal contrasts $(n \leq I - 1)$	Bonferroni significance level α/n
pairwise comparisons	Tukey method
comparison with a control group	Dunnett's method
complex nonorthogonal or complex unplanned comparisons	Scheffé: critical value $\sqrt{(I-1)F_{I-1,N-I,95\%}}$